

# Math 255B Lecture 2 Notes

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## 1 Perturbation of Fredholm Operators and The Logarithmic Law

### 1.1 Perturbation of Fredholm operators

Last time, said that  $T \in \mathcal{L}(B_1, B_2)$  is **Fredholm** if  $\dim \ker T < \infty$  and  $\dim B_2 / \operatorname{im} T < \infty$ . We were proving that the Fredholm property is preserved under small perturbations.

**Theorem 1.1.** *Let  $T \in \mathcal{L}(B_1, B_2)$  be a Fredholm operator. If  $S \in \mathcal{L}(B_1, B_2)$  is such that  $\|S\|_{\mathcal{L}(B_1, B_2)}$  is sufficiently small, then  $T + S$  is Fredholm, and  $\operatorname{ind}(T + S) = \operatorname{ind} T$ .*

*Proof.* Produce

$$\mathcal{P} = \begin{bmatrix} T & R_- \\ R_+ & 0 \end{bmatrix} : B_1 \oplus \mathbb{C}^{n_-} \rightarrow B_2 \oplus \mathbb{C}^{n_+},$$

where  $n_+ = \dim \ker T$  and  $n_- = \dim \operatorname{coker} T$ . We get that

$$\tilde{\mathcal{P}} = \begin{bmatrix} T + S & R_- \\ R_+ & 0 \end{bmatrix}$$

is invertible as well, with inverse

$$\mathcal{E} = \begin{bmatrix} E & E_+ \\ E_- & E_{-+} \end{bmatrix} : B_2 \oplus \mathbb{C}^{n_+} \rightarrow B_1 \oplus \mathbb{C}^{n_-},$$

where

$$E : B_2 \rightarrow B_1, \quad E_+ : \mathbb{C}^{n_+} \rightarrow B_1, \quad E_- : B_2 \rightarrow \mathbb{C}^{n_-}, \quad E_{-+} : \mathbb{C}^{n_+} \rightarrow \mathbb{C}^{n_-}.$$

Since  $\mathcal{E}$  is a right inverse for  $\tilde{\mathcal{P}}$ ,

$$\begin{bmatrix} T + S & R_- \\ R_+ & 0 \end{bmatrix} \begin{bmatrix} E & E_+ \\ E_- & E_{-+} \end{bmatrix} = \begin{bmatrix} * & * \\ * & R_+ E_+ \end{bmatrix}.$$

This is the identity map, so  $R_+E_+ = 1$  on  $\mathbb{C}^{n_+}$ . So  $E_+$  is injective. Similarly, we get  $E_-R_- = 1$  on  $\mathbb{C}^{n_-}$ , so  $E_-$  is surjective.

We claim that  $T + S$  is Fredholm. For the kernel, we have  $x \in \ker(T + S) \iff (T + S)x = 0$ . We can write this as

$$\tilde{\mathcal{P}} \begin{bmatrix} x \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ a_+ \end{bmatrix},$$

where  $a_+ = R_+x \in \mathbb{C}^{n_+}$ . Using the inverse  $\mathcal{E}$ , this is

$$\begin{bmatrix} x \\ 0 \end{bmatrix} = \mathcal{E} \begin{bmatrix} 0 \\ a_+ \end{bmatrix} = \begin{bmatrix} E_+a_+ \\ E_-a_+ \end{bmatrix}.$$

So we get that  $x \in \ker(T + S) \iff x = E_+a_+$  for some  $a_+ \in \ker E_-$ . So  $E_+ : \ker E_- \rightarrow \ker(T + S)$  is surjective. Since we already know  $E_+$  is injective, we get that  $\ker(T + S)$  is finite dimensional with  $\dim \ker(T + S) = \dim \ker(E_-) \leq n_-$ .

Next consider  $B_2/\text{im}(T + S)$ : Given  $y$ ,

$$\begin{aligned} (T + S)x = y &\iff \tilde{\mathcal{P}} \begin{bmatrix} x \\ 0 \end{bmatrix} = \begin{bmatrix} y \\ a_+ \end{bmatrix} \\ &\iff \begin{bmatrix} x \\ 0 \end{bmatrix} = \mathcal{E} \begin{bmatrix} y \\ a_+ \end{bmatrix}. \end{aligned}$$

So we get that  $x = Ey + E_+a_+$  and  $0 = E_-y + E_-a_+$ . We get that  $y \in \text{im}(T + S) \iff E_-y \in \text{im } E_-$ . Now consider  $B_2/\text{im}(T + S) \rightarrow \mathbb{C}^{n_-}/E_-$  sending  $y + \text{im}(T + S) \mapsto E_-y + \text{im } E_-$ . This map is surjective, as  $E_-$  is surjective, and it is also injective. So  $\dim \text{coker}(T + S) = \dim \text{coker } E_- < \infty$ .

So  $T + S$  is Fredholm, and

$$\text{ind}(T + S) = \dim \ker E_- - \dim \text{coker } E_- = \text{ind } E_- = n_+ - n_- = \text{ind}(T). \quad \square$$

**Corollary 1.1.** *The set of Fredholm operators is open in  $\mathcal{L}(B_1, B_2)$ , and  $T \mapsto \text{ind}(T)$  is locally constant.*

The proof also gives the following:

**Corollary 1.2.**  *$T \mapsto \dim \ker T$  is upper-semicontinuous on the set of Fredholm operators.*

## 1.2 The logarithmic law

**Proposition 1.1.** *Let  $T_1 \in \mathcal{L}(B_1, B_2)$  and  $T_2 \in \mathcal{L}(B_2, B_3)$  be Fredholm. Then  $T_2T_1$  is Fredholm, and we have the **logarithmic law**:*

$$\text{ind } T_2T_1 = \text{ind } T_2 + \text{ind } T_1.$$

*Proof.* Consider  $T'_1 : \ker T_2 T_1 \rightarrow \ker T_2$  sending  $x \mapsto T_1 x$ . Then  $\ker T'_1 = \ker T_1$ , so  $\dim(\ker(T_2 T_1)/\ker T_1) \leq \dim \ker T_2$ . So  $\dim \ker T_2 T_1 < \infty$ .

Now consider

$$0 \longrightarrow B_2/\operatorname{im} T_1 \xrightarrow{T'_2} B_3/\operatorname{im} T_2 T_1 \xrightarrow{q} B_3/\operatorname{im} T_2 \longrightarrow 0,$$

where

$$T'_2(x + \operatorname{im} T_1) = T_2 x + \operatorname{im} T_2 T_1, \quad q(x + \operatorname{im} T_2 T_1) = x + \operatorname{im} T_2.$$

The sequence is exact at  $B_3/\operatorname{im} T_2 T_1$ :  $\operatorname{im} T'_2 \subseteq \ker q$  by definition, and if  $x + \operatorname{im} T_2 T_1 \in \ker q$ , then  $x \in \operatorname{im} T_2$ , so  $x + \operatorname{im} T_2 T_1 \in \operatorname{im} T'_2$ . We have

$$\dim(\operatorname{coker}(T_2 T_1)/\ker q) \leq \dim \operatorname{coker} T_2, \quad \dim \ker q = \dim \operatorname{im} T'_2 \leq \dim \operatorname{coker} T_1,$$

so

$$\dim \operatorname{coker}(T_2 T_1) \leq \dim \operatorname{coker} T_1 + \dim \operatorname{coker} T_2.$$

To compute the index, consider

$$L(t) = \begin{bmatrix} I_2 & 0 \\ 0 & T_2 \end{bmatrix} \begin{bmatrix} I_2 \cos t & I_2 \sin t \\ -I_2 \sin t & I_2 \cos t \end{bmatrix} \begin{bmatrix} T_1 & 0 \\ 0 & I_2 \end{bmatrix} : B_1 \oplus B_2 \rightarrow B_2 \oplus B_3, \quad t \in \mathbb{R}, I_2 = \operatorname{id}_{B_2}.$$

This is a product of three Fredholm operators, so  $L(t)$  is Fredholm for all  $t$  and  $t \mapsto \mathcal{L}(t)$  is continuous. So  $\operatorname{ind} L(t)$  is independent of  $t$ ! When  $t = 0$ ,

$$L(0) = \begin{bmatrix} I_2 & 0 \\ 0 & T_2 \end{bmatrix} \begin{bmatrix} T_1 & 0 \\ 0 & I_2 \end{bmatrix} = \begin{bmatrix} T_1 & 0 \\ 0 & T_2 \end{bmatrix},$$

so  $\operatorname{ind} L(0) = \operatorname{ind} T_1 + \operatorname{ind} T_2$ . When  $t = -\pi/2$ , we get

$$L(-\pi/2) = \begin{bmatrix} 0 & -I_2 \\ T_2 T_1 & 0 \end{bmatrix}.$$

So

$$L(-\pi/2) \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ T_2 T_1 x \end{bmatrix},$$

which gives  $\ker L(-\pi/2) = \ker T_2 T_1 \oplus \{0\}$ . We get  $\operatorname{ind} L(-\pi/2) = \operatorname{ind}(T_2 T_1)$ . Since the index is locally constant, we get the logarithmic law.  $\square$